Is Hume's Principle Analytic?

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Abstract

Neologicism in the philosophy of mathematics is threatened by the so-called *Bad Company* problem. Neologicists aim to provide a foundation for arithmetic on the basis of Hume's Principle (HP), which states that the cardinal number of F is identical to the cardinal number of G if and only if F and G can be put into one-to-one correspondence. However, there are many principles with the same form as HP that are unacceptable, either because they are inconsistent, or because they are jointly incompatible with HP; the problem consists in distinguishing acceptable principles from unacceptable ones. In this paper I present a *pluralist* response to the Bad Company problem. Over the years, neologicists have formulated a plethora of criteria of acceptability, often to little avail; I claim that alternative criteria can all be legitimate. I finally show that pluralism about abstraction surprisingly retains the analyticity of Hume's Principle.

I present a pluralist response to the so-called *Bad Company* problem ([11], [4]; see [23] for introduction) for neologicism in the philosophy of mathematics ([37], [20]; see [13] for introduction).

Neologicists aim to provide a foundation for arithmetic on the basis of Hume's Principle (HP), which states that the cardinal number of the concept F is identical to the cardinal number of the concept G if and only if F and G can be put into one-to-one correspondence:

$$\forall F \forall G\ (\#F = \#G \leftrightarrow F \approx G) \tag{HP}$$

HP is sufficient to derive all the standard axioms of second-order Peano Arithmetic PA^2; this result is now known as Frege's Theorem ([3], [5]). However, there are many principles with the same form as HP that unacceptable, either because they are inconsistent (as Frege's own Basic Law V) or because they are jointly incompatible with HP. As remarked by [23, 324], "attractive abstraction principles such as Hume's Principle are surrounded by bad companions"; the Bad Company problem consists in sorting out acceptable principles from unacceptable ones.

Over the years, neologicists have formulated a plethora of criteria of acceptability (cf. [8] for an overview). I will claim that alternative criteria can all be legitimate. I will also show that pluralism about criteria surprisingly retains the analyticity of Hume's Principle.

In Section 2 of this paper I introduce the Bad Company problem. I will first provide a general characterisation of Bad Company as follows[1]: a case A keeps 'bad company' if there is some case B such that *i.* A and B have the same form, *ii.* A and B have the same virtues, and *iii.* A and B are rival to each other. In the case of Hume's Principle, *i.* HP and its bad companions have the same logical form; *ii.* neologicists claim that HP is analytic in virtue of its form ([41, 312]); and *iii.* HP and its bad companions cannot all be true together. Dummett's and Boolos' versions of the Bad Company problem target the analyticity of HP. The argument goes as follows: (1) Neologicists claim that HP is analytic; (2) if HP was analytic, its bad companions would also be analytic; (3) however, the bad companions cannot be analytic (if HP is), therefore HP is not analytic.

[1]The Bad Company problem is not limited to Neologicism; cf. [22], [12] and [28].

In Section 3 I consider some *criteria* proposed by the neologicists. Neologicists have formulated increasingly strong criteria for acceptable abastraction. These criteria include, among others, consistency, Field-conservativeness[2] ([39]) and Strong Stability[3] ([16], [20]). [8] recently claimed that the Bad Company problem can be solved by appeal to Strong Stability in combination with two other criteria, namely Heck-Stability[4] and monotonicity[5]. The main motivation for Strong Stability and its like is that it ensures that 'good' abstraction principles are *irenic*–i.e., roughly, (i) conservative and (ii) compatible with any other conservative principle ([41]). Irenicity is motivated in its turn by the idea that each good abstraction principle must be *unmatched*, that is, such that " there is no other abstraction incompatible with it which has exactly the same other virtues" ([20, 426]). I will argue that irenicity is neither sufficient nor necessary for unmatchedness. As regards sufficiency, I rely on a result by [26] showing that there are inconsistent classes of strongly stable principles; this shows that there may be cases in which an abstraction is unmatched but not irenic. As regards necessity, I will rely on a remark by [6] pointing out that the restriction of the quantifiers a principle to its own abstracts make the principle conservative and irenic; therefore, irenicity does not guarantees unmatchedness. I conclude that irencity is an ill-motivated criterion[6].

In Section 4 I elaborate an alternative to unmatchedness, which I will call *pluralism about criteria*. In order to solve the problem, we may need a new picture of how abstraction works under the risk of running into Bad Company. We will start with a community of speakers that adopts an abstraction principle (AP).[7] By adopting AP, the community (*i*) take AP to govern the meaning of the corresponding abstraction operator, (*ii*) have a disposition to infer following the (the right-to-left direction of) AP, and (*iii*) take AP to be true. I claim that adoption depends on the relevance of the corresponding equivalence relation to the community's goals, and it is therefore a function of the community's background theories and, more in general, of its interests. A *criterion* is just the set of abstraction principles that the community accepts[8]. At some point, the community might start theorising about the criteria themselves, hence arriving at standard conditions for acceptable abstraction. In doing so, they may rely on mathematical resources. However, having a theory of criteria is not required in order to adopt an abstraction principle. The key point is that once an abstraction principle is adopted, there are other principles that the community can accept without retraction. How does retraction work? Suppose that the community accepts two principles, but later discovers that they are incompatible (for simplicity, we shall assume that both principles are consistent). The community can simply choose one between the two principles and discharge the other. Alternatively, they may restrict the quantifiers of either of those principles. I will make this idea precise using *many-sorted languages*. A (higher-order) many-sorted language features many sorts of first-order variables, each ranging over its own universe, and many sorts of higher-order variables, ranging over subsets of those universes. Intuitively, each sort will be interpreted as the good company picked up by a given criterion. More precisely, the domain of the sort corresponding to a criterion will consist of (i) a domain of "basic", viz. non-abstract, objects, or a subset thereof; and (ii) all the abstract entities that are introduced by the abstraction principles that belong to that criterio. Pluralism about criteria is simply the view that more than one criterion can in principle be legitimate.

[2] An abstraction principle AP is *(semantically Field-)conservative* if for any theory T to which AP can be consistently added, and for any sentence ϕ of the language of T whose quantifiers have been restricted to the ontology of the original theory, $\{T + AP\} \models \phi$ only if $T \models \phi$.

[3] An abstraction principle AP is *strongly stable* only if there is a cardinal k such that AP is γ-satisfiable, namely satisfiable in a model of cardinality γ, if and only if $\gamma \geq \kappa$.

[4] An abstraction principle AP is Heck-stable iff AP is (i) strongly stable and (ii) *critically full*, i.e. if for any critical point k of AP – that is, for any k s.t. AP is k-satisfiable, there is some $\gamma < \kappa$ s.t. for any $\lambda \leqslant \gamma < \kappa$, AP is not λ-satisfiable – any model of AP of size κ contains κ abstracts of the sort characterized by AP.

[5] An abstraction principle AP is monotonic if its equivalence relation Φ is intrinsic, that is, $\Phi(F, G)$ iff $\Phi^{F \cup G}(F, G)$ for any F and G, where $\Phi^{F \cup G}$ is the result of restricting all the quantifiers of Φ to objects that fall under either F or G .

[6] An alternative solution is proposed by [24], [27] and [33], and developed by [25]. Assessing this solution is beyond the scope of this paper; cf. [34].

[7] Cf. [25] for a similar approach. In the early days of Neologicism, it was common to consider an individual subject ('Hero') rather than a community; nothing hinges on this point.

[8] A criterion need not be *maximal* (at any point, there might be different maximal sets of principles to which the criterion belongs), Moreover, the community might accept only one principle, e.g. because all the other principles the community might be willing to accept are derivable from that one principle.

In Section 5 I consider the philosophical view that is supported by pluralism about criteria, which I will call *pluralism about abstraction*. I will defend this view from five challenges.

Analyticity: neologicists claim that HP is analytic "by virtue of being determinative of the concept it thereby serves to explain" ([20, 14]). Analyticity in the neologicist sense ('analyticity$_{NL}$') is distinct from both metaphysical analyticity and from epistemic analyticity. Pluralism about abstraction is compatible with the analyticity$_{NL}$ of abstraction principles as long as principles belonging to different good companies introduces different concepts (cf. [32] for a defence of this view).

Generality: [15] claims that the conditional stating that if the number of F and the number of G exist, then those numbers are identical if and only if their concepts can be put into one-to-one correspondence, is "purely analytic", since it does not presuppose the existence of numbers. However, neologicist reply that that conditional cannot be analytic$_{NL}$, since understanding its antecedent presupposes the concept that HP is meant to introduce ([18], [38]). For the same reason, neologicists insist on a 'domain-neutral' understanding of the first-order quantifiers on the right-hand side of HP ([40]). The same objection may be directed against pluralism about abstraction, since in order to understand the what the appropriate sorts for HP consist in, one must already possess the concept of *Number*. I don't find this objection very convincing; by contrast, I will argue that sorts correspond to the (informal) 'domain-neutral' understanding, or, more precisely, to the part of it that is necessary to the aims of Neologicism.

Applicability: Frege argued that the concept of *Number*, on par with logical concepts, can be applied to any domain of objects whatsoever. Pluralism about abstraction is at odds with Frege's dictum since HP does not belong to all criteria. I will point out that universal applicability is not a mathematical necessity: to derive Frege's Theorem, it is indeed sufficient that infinitely many objects lie within the range of HP's first-order quantifiers, and those are already provided by HP itself. Moreover, there may be independent reasons for claiming that HP is not universally applicable ([9]). I will finally sketch two strategies that can be used to apply HP in a domain that does not contain numbers. First, the quantifiers of HP could be restricted to a non-mathematical domain. Second, bridge principles could be used to interpret Frege Arithmetic in the domain.

Cross-sortal identities: Are the cardinal numbers of one sorts identical to those of another? This question is similar to the so-called 'Julius Caesar' problem ([17]) and the problem of determining when two distinct abstraction principles introduce the same abstracts ([7]). Pluralism about abstraction is neutral with respect to cross-sortal identities; more precisely, cross-sortal identities are ill-formed in a many-sorted language. One is free to assign truth-values to those identities in one way or another (cf. [29], [25]). For example, *inflationists* such as [10], [19] and [30], who ascribe a metaphysical content to HP that goes beyond the one that is expressed by the material biconditional, usually solve the Caesar problem in the negative, arguing that no entity can belongs to two distinct sorts ([21]). By contrast, *deflationists* such as [1], [2] and [31] claim that cardinal numbers have no inner nature and so that Julius Caesar can be a number. Pluralism about abstraction does not settle cross-sortal identities; therefore, inflationism and deflationism are both up for grabs.

Collapse: Finally, one might argue that pluralism about abstraction collapses over monism since many-sorted languages can be reduced to one-sorted ones. his objection has been discussed in connection with various forms of ontological pluralism ([35], [36]). I will point out that the result holds only for first-order languages and that pluralism about abstraction is in no danger of collapse.

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