The Linguistic Foundations of Arithmetic

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1 Introduction

In late 1911 Wittgenstein was visiting Frege in Jena. As Wittgenstein was about to leave, he and Frege had an exchange that Wittgenstein recalled as follows:

> The last time I saw Frege, as we were waiting at the station for my train, I said to him "Don't you ever find *any* difficulty in your theory that numbers are objects?" He replied "Sometimes I *seem* to see a difficulty — but then again I *don't* see it".[1]

Which difficulty did Frege admit to sometimes seeming see? It is unlikely to have been the devastating difficulty for Logicism — Frege's *theory* of numbers — revealed by Russel's paradox. After all, having received Russell's fateful letter in late 1902, this difficulty had, at the time of Wittgenstein's visit, already been known to Frege for a good many years. Thus, Frege had no reason to only admit to seeming to see it. Perhaps, then, it was a difficulty concerning the linguistic justification of Frege's conception of numbers as *objects*, a difficulty that he would only articulate in writing several years later.

Famously, Frege sharply distinguished between *objects* and categorically distinct higher-order entities called *concepts* (or, more generally, *functions*). Given this distinction, Frege felt compelled to regard numbers as objects because number words like 'two' in arithmetical contexts like 'Two is an even number' appear to function in the same way as proper names — and, hence, as object-denoting expressions — such as, for instance, 'Sirius' in 'Sirius is a fixed star'. However, as evidenced by a diary entries from March 24th 1924, Frege had come to suspect that appearances might be deceptive:

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[1]Reported by Geach in Anscombe and Geach (1961, p. 131) .

> [O]ne comes to suspect that our way of using language is misleading, that number-words are not proper names of objects at all and words like 'number' and '[even] number' ... are not concept-words; and that consequently a sentence like '[Two] is [an even] number' simply does not express that an object is subsumed under a concept and so just cannot be construed like the sentence 'Sirius is a fixed star'. But then how is it to be construed?[2]

Frege may well have been right to be suspicious. As I will argue in this talk, from the perspective of natural language semantics, a case can be made that number words in arithmetical contexts function as concept-denoting rather than object-denoting expressions. Consequently, numbers should be conceived of as concepts rather than objects.

2 Type-Theoretic Semantics and Higher-Orderism

My argument will be developed against the background of (extensional, functional) type-theoretic semantics, widely used in contemporary semantic theorizing.[3] Type-theoretic semantics countenances two basic semantic types: t for truth-value and e for entity (or object). In addition, it countenances derivative types $\langle \rho, \sigma \rangle$ as licensed by the following rule: ρ and σ are types iff $\langle \rho, \sigma \rangle$ is a type. Expressions of derivative types denote *functions*. Functions whose values are truth-values are *concepts*. Concepts whose arguments are objects are *first-level* concepts. Concepts whose arguments are nth-level concepts are n+1*th-level* concepts. Finally, type-theoretic semantics countenances at least the following three principles of semantic composition. Functional Application (FA), Predicate Modification (PM), Predicate Abstraction (PA).[4]

Although similar to Frege's own views in many ways, standard type theoretic semantics as practiced in contemporary linguistics departs from the Frege in one crucial respect. Standard type theoretic semantics treats higher-order entities — i.e. the denotations of derived functional types $\langle \rho, \sigma \rangle$ — as *sets*. However, for Frege sets were themselves objects and therefore not categorically distinct from the latter. Thus, the functions of standard type theoretic semantics are not genuinely higher-order entities.

Under the label of (genuine) *Higher-Orderism*, recent years have seen a renaissance of Frege's views that stresses the importance for ontological theorizing of taking Frege's categorical distinction between objects and higher-level entities seriously.[5] Methodologically, I will thus appeal to both of these approaches. Since the focus is on the semantic function of number words

[2] Frege (1979, p. 263).
[3] See Heim and Kratzer (1998) for an introduction.
[4] Unless explicitly indicated, the operating composition rule will be FA.
[5] For a recent overview, see Skiba (2021).

in natural language, I will freely avail myself of standard type theoretic semantics and its set-theoretic framework. At the same time, since I intend my conclusions to be ontologically significant, I will regard this set-theoretic framework as a mere heuristic[6], that — albeit technically convenient — can and should ultimately be dispensed with in favour of a genuinely higher-order framework.

3 Number Words: Three Uses, One Function

Number words in natural language can be used in a variety of different ways. Thus, consider the number word ‘two’ as it is used in the following sentences:

(1)	a.	Mars has *two* moons.	*quantificational use*
	b.	The number of Mars's moons is *two*.	*specificational use*
	c.	*Two* is an even number.	*numeral use*

Do these different uses correspond to different semantic functions? Or can number words be understood semantically uniformly—i.e. as performing the same semantic function—in all of these different uses? As per the second option, I will argue that ‘two’ in all three of these uses performs the same function, viz. that of a concept-denoting expression.[7]

3.1 The Quantificational Use

As for the semantic function of ‘two’ in its quantificational use in (1a), I will follow current linguistic theory in treating ‘two’ as a first-level expression — i.e. as a predicational adjective much like, say, ‘interesting’ in ‘Mars has interesting moons’ — and that (1a) is to be analysed as follows:

[6]In the sense of Wright (2007).

[7]For recent decent, see Snyder et al. (2022). Thus, my proposal is a species of of what is sometimes called ‘the adjectival strategy’ in the philosophy of mathematics, see e.g. Hodes (1984) and, after a fashion, Hofweber (2005a,b). Unlike the former, I am interested in whether this strategy is viable as an interpretation of natural language number talk. And adding to the latter, I aim to strengthen the case that it is thus viable.

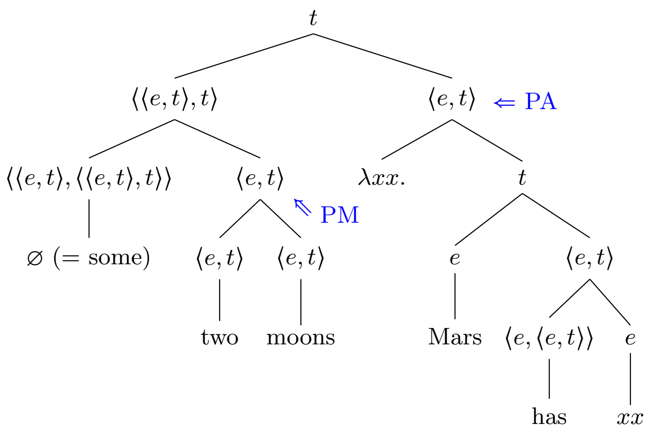

Figure 1: Compositional analysis of (2)

This analysis rests on two assumptions. First, that in (2), the number word 'two' is a (plural) type $\langle e, t\rangle$ expression that denotes the *first-level number concept* of being two, a concept that subsumes all and only those type e pluralities of (exactly) two.[8] Second, that 'two moons' is the complement of an empty determiner which can be interpreted as the existential determiner 'some'.[9] Given this analysis, the logical form of (1a) is:

(1a*) $\exists xx(\texttt{two}(xx) \wedge \texttt{moons}(xx) \wedge \texttt{has}(\texttt{Mars}, xx))$.

3.2 The Specificational Use

In his *Grundlagen*, Frege (1884, §57) famously proposed that a sentence like (1b) be analysed as a first-order identity statement, i.e. as a sentence in which the *the-number-of* term 'the number of Mars's moons' and the number word 'two' function as (singular) type e expressions and 'is' functions as a type $\langle e, \langle e, t\rangle\rangle$ expression that denotes type e identity, i.e. identity between objects. Recent years have seen the advent of anti-Fregean analyses of (1a) who are united in the claim that the number word 'two' is not a type e expression but rather has the same, concept-denoting semantic function as it has in (1a).[10]

[8] Plural type e expressions are treated as denoting genuinely plural objects or pluralities understood as mereological sums and plural $\langle e, t\rangle$ expressions as denoting concepts that subsume pluralities so understood; see Link (1983). On Link's treatment, apparently plural reference and predication is, at bottom, singular reference to and singular predication of sums. There is also an alternative view according to which plural referring expressions and plural predicates plurally refer and apply to several non-plural objects at once. For a recent treatise on this issue, see Florio and Linnebo (2021).

[9] See Ionin and Matushansky (2006, 322) by way of Krifka (1999).

[10] See, e.g., Hofweber (2005b); Moltmann (2013); Felka (2014).

Although not without merits, the extant anti-Fregean analyses of (1b) are unsatisfactory.[11] In my talk, I thus develop a novel and improved anti-Fregean analysis of (1b). According to this analysis, Frege was right to analyse (1b) as an identity statement but was mistaken about its level: the identity in question concerns first-level (Fregean) concepts rather than objects.[12]

This analysis of (1b) is predicated on the following analysis of *the-number-of* terms according to which they are higher-order descriptions that denote number-concepts:

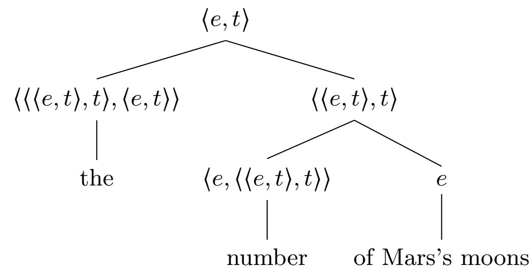

Figure 2: Proposed compositional analysis of *the-number-of* terms

This analysis rests on three main tenets. First, that 'of Mars's moons' in (1b) is a plural type e expression that denotes the plurality of all of Mars's moons. Second, that 'the' in (1b) functions as a type $\langle\langle\langle e,t\rangle,t\rangle,\langle e,t\rangle\rangle$ expressions that denotes a higher-order description function which maps any second-level concept to the unique first-level concept it subsumes (if there is one, otherwise it will be undefined). Third and most importantly, that 'number' in (1b), is a relational noun of type is a relational noun that functions as a type $\langle e,\langle\langle e,t\rangle,t\rangle\rangle$ expression that denotes a function that maps type e pluralities xx to the second-level concept of being an a first-level number concept under which xx fall. Consequently, 'the number of Mars's moons' functions as a higher-order definite description that denotes the unique first-level number concept that subsumes the moons Mars has. Since Mars has (exactly) two moons, it thus denotes the same first-level number concept as the one denoted by 'two'. If so, (1b) is true just in case 'the number of Mars's moons' and 'two' denote the same first-level number concept. Roughly, this means that that the logical form of (1b) is the following:[13]

[11] See e.g. Schwartzkopff (2016, 2022).

[12] Both Moltmann (2013) and Felka (2014) contend that sentences like (1b) are so-called *specificational sentences*. Today's proposal is compatible with this view, see Schwartzkopff (2022).

[13] 'X' is a variable of type $\langle e,t\rangle$, the superscripted '2' indicates that 'number' in (1b) is a relational noun. '$\equiv$' expresses identity between first-level concepts and can be defined by

(1b*) $[\iota X.\,\mathtt{NUMBER}^2(X, \mathtt{Mars's\ moons})] \equiv \mathtt{two}$.

3.3 The Numeral Use

At the end of his diary entry quoted above, Frege despaired at how to construe a sentence like (1c) ‘Two is an even number’ if it is not be construed as a first-order predication, i.e. not as expressing that an object is subsumed under a (first-level) concept. Answering Frege’s question, I propose to analyse (1c) as a higher-order predication that subsumes a first-level concept (viz. the number concept of being two denoted by ‘two’) under a second-level concept (viz. the concept of of being an even number concept denoted by ‘is an even number’). On this proposal, the logical form of (1c) is the following:[14]

(1c*) $[\lambda X.\,(\mathtt{EVEN}(X) \wedge \mathtt{NUMBER}^1(X))](\mathtt{two})$.

4 Linguistic Viability

Are the proposed higher-order interpretations of (1b) and (1c) linguistically viable? To answer this question affirmatively, much would be gained if it could be shown that the proposed higher-order interpretations of relational ‘number’ in (1b) and non-relational ‘number’ and ‘even’ in (1c) are independently motivated. I contend that they are.

As for the proposed interpretation of non-relational ‘number’ and ‘even’ in (1c), consider the fact that (1a) ‘Mars has two moons’ (logically) entails:

(2) Mars has an even number of moons.

I contend that the most plausible compositional analysis of (2) that delivers the correct truth-conditions and also accounts for the fact that (2) is a logical consequence of (1a) treats ‘an even number’ in (2) as a higher-order quantifier that restrictedly ranges over even first-level number concepts. In particular, I contend that this analysis renders ‘number’ in (2) as a *non-relational* noun that functions as a type $\langle\langle e,t\rangle,t\rangle$ expression that denotes the second-level concept of being a first-level number concept, i.e. that second-level concept that subsumes all and only the first-level number concepts. And this, of course, is precisely the interpretation of ‘number’ that is required for the proposed interpretation of (1c) as a a higher-order predication.[15]

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way of a higher-order analogue of Leibniz’ Law, where ‘$\mathbf{Z}$’ is a variable of type $\langle\langle e,t\rangle,t\rangle$: $X \equiv Y \leftrightarrow \forall \mathbf{Z}(\mathbf{Z}(X) \leftrightarrow \mathbf{Z}(Y))$.

[14]The superscripted ‘1’ indicates that ‘number’ in (1c) is a non-relational noun. More on this presently.

[15]Similarly, *mutatis mutandis*, for ‘even’.

In more detail, my proposal concerning (2) is the following. First, ‘an even number’ in (2) is to be analysed as follows:

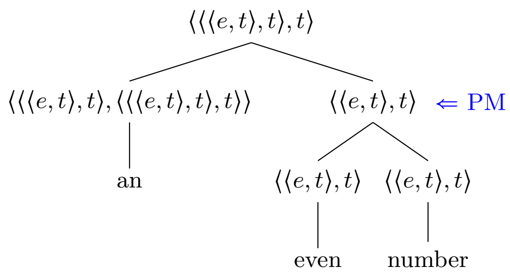

Figure 3: Compositional analysis of ‘an even number’ in (2)

Second, (2) itself is to be analysed as:

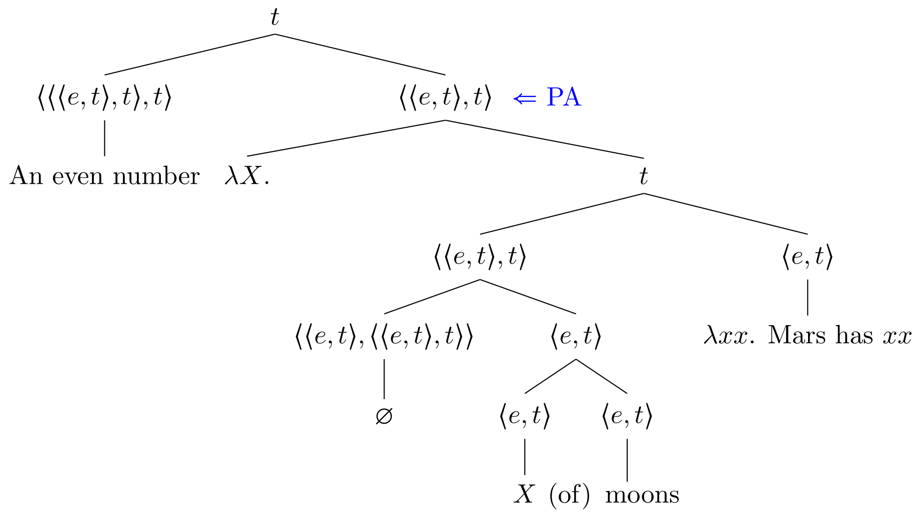

Figure 4: Compositional analysis of (2)

And third, given this analysis, the logical form of (2) is:

(2*) $\exists X(\texttt{EVEN}(X) \wedge \texttt{NUMBER}^1(X) \wedge \exists xx(X(xx) \wedge \texttt{moons}(xx) \wedge \texttt{has}(\texttt{Mars}, xx))$.

The case for the linguistic viability of the proposed higher-order interpretation of relational ‘number’ in (1b) is now the following. First, the independently motivated interpretation of ‘number’ in (1c) as a non-relational noun of type $\langle\langle e, t\rangle, t\rangle$ that denotes the second-level concept of being a first-level number concept can be defined as the existential closure of its proposed relational type $\langle e, \langle\langle e, t\rangle, t\rangle\rangle$ interpretation in (1b), but not vice versa:

$\texttt{NUMBER}^1 = \lambda X.\ \exists xx\ \texttt{NUMBER}^2(X, xx)$. That is, to be a number concept is to stand in the number relation to a plurality of objects.[16] Second, on this proposal, relational 'number' in (1b) and non-relational 'number' in (1c) bear a striking resemblance to the two uses of the noun 'mother' as it respectively occurs in:

(3) a. the mother of Philipps's children.
b. Elizabeth is a mother.

For similar to 'number' in (1b), 'mother' in (3a) is a relational noun, i.e. it functions as a type $\langle e, \langle e, t\rangle\rangle$ expression that denotes a function that takes objects xx to the first-level concept of being xx's mother. Similar to 'number' in (1c), 'mother' in (3b) is a non-relational noun, i.e. it functions as a type $\langle e, t\rangle$ expression that denotes the first-level concept of being a mother. And just like non-relational 'number' can be defined as the existential closure of relational 'number', non-relational 'mother' can similarly be defined in terms of relational 'mother', but not vice versa: $\texttt{mother}^1 = \lambda x.\ \exists xx\ \texttt{mother}^2(x, xx)$. That is, to be a mother is to stand in the mother relation to some plurality.

This one-way definability is the hall-mark of relational nouns and their non-relational counterparts.[17] Thus, the fact that i) the non-relational higher-order interpretation of 'number' in (1c) is — by way of (2) — independently motivated and ii) can be defined in terms of the proposed higher-order relational interpretation of 'number' in (1b) lends credence to the linguistic viability of this proposal.

References

Anscombe, Elizabeth and Peter Geach 1961: *Three Philosophers: Aristotle, Aquinas, Frege*. Ithaca, NY: Cornell University Press.

Barker, Chris 2011: 'Possessives and relational nouns'. In von Heusinger, Klaus, Claudia Maienborn and Paul Portner (eds.): *Semantics: An International Handbook of Natural Language Meaning*, vol. 2, Berlin: de Gruyter, pp. 1009–1030.

Bylinina, Lisa and Rick Nouwen 2018: 'On "Zero" and Semantic Plurality'. *Glossa: a journal of general linguistics* 3(1).

Felka, Katharina 2014: 'Number Words and Reference to Numbers'. *Philosophical Studies* 168(1), pp. 261–282.

[16] As formulated, this means that the empty first-level concept — i.e. the concept of being zero — is not a number concept. For a plausible account of how to treat the concept of being zero as a number concept, see Bylinina and Nouwen (2018).

[17] See e.g. Barker (2011).

Florio, Salvatore and Øystein Linnebo 2021: *The Many and the One: A Philosophical Study of Plural Logic*. Oxford: Oxford University Press.

Frege, Gottlob 1884: *Die Grundlagen Der Arithmetik*. Breslau: Verlag W. Koebner.

—— 1979: *Posthumous Writings*. Oxford: Blackwell.

Heim, Irene and Angelika Kratzer 1998: *Semantics in Generative Grammar*. Oxford: Blackwell.

Hodes, Harold T. 1984: 'Logicism and the Ontological Commitments of Arithmetic'. *Journal of Philosophy* 81(3), pp. 123–149.

Hofweber, Thomas 2005a: 'Number Determiners, Numbers, and Arithmetic'. *Philosophical Review* 114(2), pp. 179–225.

—— 2005b: 'A Puzzle about Ontology'. *Nous* 39(2), pp. 256–283.

Ionin, Tania and Ora Matushansky 2006: 'The Composition of Complex Cardinals'. *Journal of Semantics* 23(4), pp. 315–360.

Krifka, Manfred 1999: 'At Least Some Determiners Aren't Determiners'. In Turner, Ken (ed.): *The Semantics/Pragmatics Interface from Different Points of View*, vol. 1, New York, NY: Elsevier, pp. 257–291.

Link, Godehard 1983: 'The Logical Analysis of Plurals and Mass Terms: A Lattice-theoretical Approach'. In Bäuerle, Rainer, Christoph Schwarze and Arnim von Stechow (eds.): *Meaning, Use, and Interpretation of Language*, Berlin: de Gruyter, pp. 303–323.

Moltmann, Friederike 2013: 'Reference to Numbers in Natural Language'. *Philosophical Studies* 162(3), pp. 499–536.

Schwartzkopff, Robert 2016: 'Number Sentences and Specificational Sentences: Reply to Moltmann'. *Philosophical Studies* 173(8), pp. 2173–2192.

—— 2022: 'How Not to Analyse Number Sentences'. *Philosophia Mathematica* 30(2), pp. 200–222.

Skiba, Lukas 2021: 'Higher-order Metaphysics'. *Philosophy Compass* 16(10).

Snyder, Eric, Richard Samuels and Stewart Shapiro 2022: 'Resolving Frege's Other Puzzle'. *Philosophia Mathematica* 30(1), pp. 59–87.

Wright, Crispin 2007: 'On Quantifying into Predicate Position'. In Leng, Mary, Alexander Paseau and Michael Potter (eds.): *Mathematical Knowledge*, Oxford: Oxford University Press, pp. 150–174.