Hilbert’s Early Metatheory Revisited: Categoricity and Interpretability

David Hilbert’s early work on the foundations of mathematics is mainly known for its formal axiomatic approach. In his influential monograph *Foundations of Geometry* from 1899, he provided one of the first abstract axiomatizations of elementary Euclidean geometry. In a similar manner, he laid down the first system of formal axioms for analysis or the arithmetic of the real numbers in the classic investigation “On the concept of number” (1900). A central innovation of these formative contributions to modern axiomatics was the role assigned to metatheoretical issues. In particular, Hilbert carried out systematic investigations of the independence, consistency, and completeness of his axiom systems and achieved many important metatheoretical results that contributed to building of new solid foundations for these mathematical theories.

From a historical point of view, an interesting problem is to explain how these metatheoretical notions were first introduced and conceptualized in the early stages of modern axiomatics. In particular, the notion of categoricity plays a key role in this context, for it is closely connected to Hilbert’s remarkable axiom of completeness (henceforth AC) in analysis and geometry. One immediate consequence of the inclusion of this axiom, both in the geometrical and arithmetical contexts, was that it rendered the respective axiom systems categorical. Nevertheless, during this early period, Hilbert seemed to have established this important consequence and the metatheoretic property of categoricity only vaguely and informally. For instance, in connection to the axiom system for Euclidean geometry, he restricted himself to the brief observation that the geometry obtained by the admission of AC “is thus none other than the ordinary space Cartesian geometry in which the [line] completeness axiom V.2 also holds” (Hilbert 1903, p. 59, our emphasis). The schematic character of Hilbert’s remarks has been emphasized by Awodey and Reck (2002), in their important paper on the history of the concepts of completeness and categoricity: “(...) asserting simply and unequivocally that Hilbert understand his axioms [for Euclidean geometry] to be categorical would be too strong. Note that, like Dedekind, he does not yet work with an explicit, general notion of isomorphism in *Grundlagen*. Beyond that, he does not state a theorem that establishes, even implicitly, that his axioms are categorical; he leaves it at the short remarks above, without proofs. (ibid., p. 11.) Thus, according to this dominant assessment, Hilbert’s did not have a precise understanding of the model-theoretic concepts of isomorphism and categoricity at the time of Foundations.

The central aim of this talk will be to provide a historically sensitive discussion of Hilbert’s early metatheory of formal axiomatics and to re-assess its role in the development of a model-theoretic conception of theories. We undertake this task by closely examining Hilbert’s published work as well as his detailed notes for lecture courses on the foundations of mathematics, corresponding to this early period. We will argue that it is possible to obtain a more refined historical picture of his early metatheory of axiom systems, and particularly of his understanding of categoricity, when these important unpublished sources are taken into account. Specifically, the discussion will focus on two interpretative points. The first concerns Hilbert's conceptual understanding of mathematical languages and their interpretations. Based on the analysis of unpublished material, we will contend that this understanding was fundamentally informed by a particular notion of isomorphism. Roughly, Hilbert conceived isomorphism as the existence of a bijective mapping between two mathematical systems that "preserves" relational structure. This means that his characterization of isomorphisms only considered mappings between "models" of the axioms of a mathematical theory. Thus, unlike our modern "language-based" concept of isomorphism, Hilbert's early notion was essentially theory-based or theory-dependent. This theory-based conception of isomorphism naturally had consequences for Hilbert's logical understanding of the "interpretation" of a (formal) mathematical language. In this regard, we will claim that the concepts of a translation between two mathematical languages and the interpretation between theories turn out to be particularly illuminating for the reconstruction of Hilbert's early semantic views. Specifically, we will argue that it is possible to distinguish between two notions of translations of mathematical languages in his early metatheory of formal axiomatics. We will call the first notion "Dedekind translation" due to its conceptual background in Dedekind's metatheoretical work, particularly in *Was sind und was sollen die Zahlen?* (1888). Briefly, this kind of translation is a mapping between two (full interpreted) languages of the same form such as the logical form *and* the grammar and preserved. In contrast, we will argue that Hilbert's metatheoretical work on geometry, especially his consistency and independence proofs, is grounded on a different conception of a translation between mathematical languages, which is closer to the modern model-theoretic notion. Schematically, in these translations logical form but not grammar is preserved. We dubbed this kind of translations "Hilbert's translations". We make this conception of translation more precise by appealing to the modern notions of interpretability of a structure into another structure, inner models, and semantic interpretation of a theory into another theory.

The second interpretative point regards Hilbert's understanding and investigations on the categoricity of his axiom systems. As mentioned, his reflections on categoricity in published sources are limited to some informal and very schematic remarks. This especially applies to the case of his axiom system for analysis. Nevertheless, we have located a detailed discussion of this topic in the notes for the lecture course *Logische Prinzipien des mathematischen Denkens* from 1905. Hilbert's discussion also includes a sketchy proof of the categoricity of the axioms for the real numbers. We provide a detailed reconstruction of this proof-sketch, including a comparison with Dedekind's famous categoricity result in *Was sind und was sollen die Zahlen?* (1888). We contend that the previous discussion of isomorphism proves relevant for assessing the logical background of Hilbert's understanding of categoricity in this early period.

References

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