Veronese's Investigation of the Archimedean Axiom and the Foundations of non-Archimedean Geometry

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Abstract[*]

The question concerning the historico-philosophical underpinnings and the foundations of non-Archimedean geometry in general, as well as Giuseppe Veronese's (1854–1917) contributions in particular, has received relatively little attention within contemporary scholarship.[1] This is somewhat surprising since Veronese's main work, the *Fondamenti di Geometria a più dimensioni e a più specie di unità rettilinee* (1891), was initially widely discussed by leading mathematicians and philosophers (i.a. Cantor, Peano, Klein, Hilbert, Poincaré, Hahn, Peirce, Brouwer, Cassirer, Natorp, etc.). Parallel with the disinterest in the subject among philosophers of mathematics, we can also detect a lack of sufficiently detailed historical analyses which would help us to better understand the relevant broader context within which non-Archimedean geometry appeared in the 1890's. Namely, even though the great reformation of mathematics initiated in the 19th century by Bolzano and Weierstraß, and then "brilliantly followed" by Dedekind, Cantor, Frege, Peano (and his school), Russell and Whitehead, and Hilbert (and his school) has been extensively researched, Veronese's role in the mathematico-philosophical debates concerning the foundations of mathematics has yet to be seriously investigated. This also goes for other pioneers of non-Archimedean approaches to mathematics such as, e.g. Paul du Bois-Reymond (1831–1899), Otto Stolz (1842–1905), Rodolfo Bettazzi (1861–1941), and Tullio Levi-Civita (1873–1941). Historiography of non-Archimedean mathematics much too often tends to either omit or misinterpret these important 19th century developments which, accordingly, leads to a lacunar and an oversimplified presentation in which Robinson's non-standard analysis from the 1960's ends up being the only "real" successor of Leibniz's infinitesimal calculus. One of our main tasks in the present talk will be to address this lacuna and to show how Veronese's and other 19th century infinitesimalists' results paved the way for most if not all major 20th century developments in non-Archimedean mathematics.

The talk will consist of two main sections. First, we shall reconstruct the historico-philosophical context within which Veronese introduced non-Archimedean geometry (Section 1), and then, secondly, we shall provide an analysis of the peculiar way in which he approached the matter of its foundations (Section 2).

Motivated by a novel trend in contemporary philosophy of mathematics—researching and engaging with historical case-studies—in Section 1, we shall primarily focus on the foundational crisis of mathematics from the 1870's known as the "crisis of intuition" and the ensuing Weierstraßian response to it which led to the establishment of the so-called "Cantor-Dedekind academic dogma" as

[*] *Note*. We have decided to leave out exhaustive references from the main body of the text; instead, some of the most important references are listed in the end.

[1] With the exception of the pioneering research on non-Archimedean mathematics by Laugwitz, Fisher, Bell, and Ehrlich. When it comes to Veronese, one should mention the books by Bussotti and Cantù.

the received orthodoxy in both mathematics proper as well as in contemporary mathematical philosophy.

In Section 2, it will be shown that Veronese—continuing the Italian tradition of studying the independence of axioms in geometry (especially Beltrami's work on non-Euclidean geometries)—was *the first* mathematician to systematically study the consequences of the independence of the Archimedean axiom in his seminal paper on the linear continuum from 1889, pre-dating the publication of the *Fondamenti*. In it, Veronese argued for the independence of the Archimedean axiom from the axiom of continuity, i.e. he claimed that there is more to continuity than what is implied by mere archimedicity, and went on to formulate and specify the so-called Veronese's absolute and relative continuity condition. We shall also see how these results were further elucidated and elaborated in the *Fondamenti* and in subsequent publications from the 1890's and early 1900's. Focusing on the *Fondamenti*, it will be shown how Veronese "took a modernist stance" not only by introducing a non-Archimedean continuum (i.e. a maximally inclusive non-Archimedean ordered field), but specifically by developing his non-Archimedean geometry in a "perfect pre-Hilbert style". We shall illustrate this by presenting some of Veronese's proto-model-theoretical insights, as well as his understanding of the notion of mathematical existence or possibility (typically but wrongly associated only with Hilbert). However, we shall also present the key philosophical tenets of what Veronese himself took to be the most important portion of his geometry – his *rectimétrie*, i.e. the theory of the so-called *intuitive linear continuum*. It will be shown that in this regard Veronese (*contra* Cantor's pointillism) was adhering to the classical Aristotelian tradition of treating the continuum as composed not of points but of one-dimensional segments of indeterminate (finite, infinitesimal or infinite) length.[2]

Finally, after reconstructing the philosophically interesting "great struggle" between the anti-infinitesimalist Cantor and infinitesimal-friendly Aristotelian Veronese, as well as the Vivanti-Bettazzi debate concerning the existence of infinitesimals which inaugurated Peano's *Rivista di Matematica*, we shall conclude the talk by pointing out some of the factors which negatively influenced the reception history of Veronese's non-Archimedean geometry, focusing specifically on Poincaré's double mistake in interpreting some key features of Veronese's theory, as well as the infamous Cantor-Peano alleged proofs against the possibility of infinitesimals. It will be shown that Poincaré-style misinterpretation is at the heart of most contemporary historical misconceptions of Veronese's theory.

Key words: Giuseppe Veronese, Archimedean axiom, non-Archimedean geometry, infinitesimals, continuum (absolute *vs.* relative).

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[2] In this part of the talk, we shall also mention the almost unknown yet very illuminating studies of Veronese by W. S. Contro and R. Peiffer-Reuter which were pointed to us by Siegmund Probst and Marco Panza.

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