The Omega Rule and the Categoricity Problem

Abstract

The omega rule is familiar enough: it allows one to move from the premises

φ(a), φ(b), φ(c), φ(d), …

to the conclusion

∀xφ(x)

As Tarski pointed out long ago, the rule is intuitively valid: if a, b, c, d, … are the objects in a denumerably infinite domain and each of these objects is φ, then every object in the domain is φ. However, the rule is not recursive: no standard computing device can determine, in a finite amount of time, whether a putative instance of the rule is genuinely an instance. It is questionable, therefore, whether finite beings like us can follow the omega rule.

Warren, though, has argued that ordinary speakers can and do follow that rule (2020, 2021). This fact, he claims, has key metasemantic consequences: it allows one to provide an inferentialist-friendly account of the determinacy of our mathematical theories. Indeed, Garson (2013) and Brîncuş (2022) have recently made a further metasemantic claim: that our (supposed) ability to follow the omega rule plays an essential role in explaining the categoricity of the quantifiers. As Garson and Brinçus point out, Carnap (1943) not only exhibited deviant interpretations of the propositional connectives – this is Carnap's Categoricity Problem, to which a number of solutions have been proposed (Carnap 1943, Smiley 1996, Garson 2013, Bonnay & Westerståhl 2016, Murzi & Topey 2021) – he also exhibited deviant interpretations of the quantifiers, which have been virtually ignored by philosophers and logicians alike. Crucially, Garson and Brîncuş claim that Carnap's deviant interpretations of the quantifiers, as well as other deviant interpretations identified by Garson, can be ruled out only on the assumption that we can follow the omega rule. As a result, Brîncuş maintains, the account of the categoricity of the quantifiers proposed by Murzi & Topey (2021), according to which the rules for the quantifiers are entirely standard, is hopeless.

Here we argue that the key idea shared by Warren, Garson, and Brîncuş – i.e., that in order to give an inferentialist-friendly account of the determinacy of our logical and mathematical language, we *must* accept that infinitary rules like the omega rule have a role to play in governing our use of that language – is mistaken. For one thing, *pace* Warren, Garson, and Brîncuş, the followability of the omega rule has no essential role to play in explaining either the categoricity of the quantifiers or the determinacy of arithmetic. And for another, from a naturalistic perspective, appealing to infinitary rules is likely to be of no help anyway, since there's no good reason to think we can follow such rules.

To argue for the first of these claims, we begin by showing that the inferentialist-friendly account of the categoricity of the quantifiers (of finite order) provided by Murzi & Topey (2021) is robust enough to rule out both Carnap's and Garson's deviant interpretations. Our crucial point is that on the *local* conception of validity we favor – i.e., a conception on which validity consists in the preservation of sequent satisfaction, where sequents may contain open formulas in addition to sentences – the validity of a (standard) formulation of ∀-I that allows one to move from the premise sequent

Γ ⊢ φ (where x doesn't appear free in Γ)

to the conclusion sequent

Γ ⊢ ∀xφ

is, despite Brîncuş's arguments to the contrary, sufficient to rule out both Carnap's and Garson's deviant interpretations of the universal quantifier. Consider, for example, Carnap's deviant interpretation on which '∀xP(x)' is true just in case every individual is a P and (in addition) b is a Q. In order for this interpretation to be compatible with the local validity of ∀-I, there must be no valuation that accords with the

interpretation but fails to make ∀-I satisfaction-preserving. Suppose, though, that in some valuation v, every object is a P but b isn't a Q. Insofar as v accords with Carnap's deviant interpretation, '∀xP(x)' is *false* in v. But notice that v satisfies the sequent '∅ ⊢ Px', since every object – i.e., every possible value of x – is a P. (This result requires that we stipulate, as usual, that a sequent containing open formulas is satisfied by a valuation just in case it's satisfied on *all* variable assignments.) So the local validity of ∀-I guarantees that v satisfies '∅ ⊢ ∀xPx' as well. But this is just to say that the local validity of ∀-I guarantees that '∀xP(x)' is *true* in v, contrary to what Carnap's deviant interpretation implies. So the local validity of ∀-I does indeed rule out that deviant interpretation.

Having shown that the account provided by Murzi & Topey (2021) does rule out all deviant interpretations of the quantifiers, we complete this part of our argument by explaining that this result makes available an inferentialist-friendly account of the determinacy of our mathematical theories: the categoricity of second-order arithmetic is an immediate consequence (via Dedekind's categoricity theorem) of the categoricity of the second-order quantifiers, and determinacy follows from categoricity, despite Warren's mistrust of model-theoretic results.

Finally, in order to argue for the second of the claims mentioned above – i.e., that, from a naturalist perspective, there's no good reason to think we can follow infinitary rules like the omega rule – we first point out that the literature's only real argument for the followability of such rules is one given by Warren, which purports to exhibit a case in which we're actually disposed to use the omega rule to infer a generalization from infinitely many instances, and then we show that this argument, despite its ingenuity, is unconvincing. The case Warren presents is one on which, plausibly, we are indeed disposed to accept both a universal claim and its infinitely many instances; the question is just whether we're disposed to accept the generalization *on the basis of* the instances or the other way around. Warren argues that we accept the generalization on the basis of the instances: there are possible beings who accept the instances (on the same grounds we do) but don't accept the generalization, and so, Warren insists, the explanation of why we accept the generalization and they do not can only be that we follow the omega rule and they do not. But as Nyseth (2021) points out, an alternative explanation, one that doesn't require any appeal to infinitary rules, is available here. And we show that Nyseth's explanation, not Warren's, is correct: it turns out that there's *conclusive* reason to think that, in the case Warren presents, we're in fact reasoning from the generalization to the instances and so aren't using the omega rule (or any other infinitary rule) at all.

In short, we show in this paper that, if our goal is to give a naturalist account of logical and mathematical determinacy, we need not appeal to infinitary rules like the omega rule, nor is doing so likely to help.

References

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