# A MINIMALIST APPROACH TO FOUNDATIONS OF (CONSTRUCTIVE) MATHEMATICS

In classical mathematics, Zermelo-Fraenkel set theory emerged as the standard foundational theory after the crisis of foundations of mathematics. However for those mathematicians which decided to follow a constructive approach, no theory turned out to be a standard accepted by the clear majority of the community.

Maietti and Sambin [MS05] distinguish between two main approaches: "One maintains that the meaning of mathematics lies in its computational content, and thus keeps its formalization in a computer language in mind. It is usually associated with type theory [...]. The other favours the mathematical structure beyond its particular presentations. It is usually expressed through category theory and often identified with topos theory". This distinction has its roots in the contrast between the intensional and extensional view of mathematics.

Bridges [Bri98] instead distinguishes between Brouwerian intuitionism and Russian recursive constructive mathematics. These two approaches agree on the choice of intuitionistic logic, however they accept mathematical principles which sometimes are not compatible each other (and sometimes also incompatible with classical mathematics CLASS). Bridges refers to Bishop's mathematics (there called BISH) as the "third variety" of constructivism and claims that "there is a strong case for regarding BISH as the constructive core of mathematics, since every theorem of BISH is also a theorem of INT, RUSS, and CLASS". However this claim is not a (meta)theorem, since BISH is not a fully determined formal system. Instead it is used as a sort of criterium to exclude some mathematical principles from BISH (those principles which are false in INT, RUSS or CLASS). In fact, a precise formulation of BISH as a formal system was never given, neither this was in the spirit of Errett Bishop's approach, which used to adopt the attitude of the working mathematician (of course faithful to his constructive understanding of mathematics).

An attempt to define a precise common core foundational theory for constructive mathematics which could answer the compatibility requirements raised by Maietti and Sambin [MS05] and by Bridges [Bri98] is the formal system of the Minimalist Foundation **MF** introduced by Maietti in [Mai09]. Such a theory is based on variants of Martin-Löf type theory and consists of two levels: one intensional and another extensional. The former is the level for the computable intensional content of mathematics, while the latter is the *user-friendly* extensional level where ordinary mathematics lives. The two levels are connected since the latter can be interpreted in the former by means of a setoid model as shown in [Mai09]. Thus the so-called *setoid hell* is avoided by presenting the extensional level, an abstract account of it, as part of the system (and not as a derived construction). This division in levels reconciles the two approaches described in [MS05] by abstractly maintaining both of them, but in two distinct, although communicating, parts of the theory. Concerning the compatibility desired by Bridges, each level of the Minimalist Foundation is designed in such a way to keep distinctions between logic and mathematics, and between two different degrees of complexity, obtaining a theory which turns out to be compatible (once the adequate level is chosen) with the main classical and intuitionistic, predicative and impredicative, foundational theories in the literature.

Of course, one concern about a candidate core foundational theory is its ability to support the main mathematical constructions; in other words the core must be small enough to be compatible with the different views on constructivism, but expressive enough to be mathematically meaningful. In this sense, the system **MF** can be easily extended with rules allowing the definition of inductively generated formal topologies (see [CSSV03]) and, eventually, with weak choice principles allowing the development of mathematical analysis.

As a last observation, we notice that in apparent contrast with the word *minimalist*, the system **MF** appears as a large list of rules involving many constructors. However, this is necessary if one wants to be compatible with many different approaches (and at the same time mathematically meaningful). Following Sambin [Sam19] we can express this with the motto "Minimalist in assumptions, maximal in conceptual distinctions."

In my talk I would like to present the features of **MF** and the main (meta)mathematical results obtained concerning its compatibility with other theories and fundamental principles like choice principles and the formal Church's thesis.

## References


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