Sheaves for Intuitionistic Logic

Somewhat simplified, the 'traditional' intuitionistic conception of logic evaluates the judgement 'that p' (or more explicitly 'that $\Box p$') not as "p is true" but as "p is verifiable" or "there is a proof of p". This talk is concerned with finding an adequate *formal* semantics that governs the interpretation of the logical constants under this conception. It has two parts. Based on historical writings and on systematic grounds, I will first review four desiderata that such a semantics has to fulfil.[1] I will then observe that none of the usual semantics for intuitionsitic logic manages to account for all four of them. In the second part, I will then introduce a sheaf semantics for intuitionistic logic, for which I argue that it does. This semantics differs in some respects from already known sheaf models. But even though it is more complex than the semantics that are more familiar to philosophers, each of its components permits an intuitive understanding.

Part 1

Arguably, the intended interpretation of the logical constants is given by the BHK clauses, which are meant to capture the notion of proposition as it is tied to the notion of verifiability (cf. Heyting 1931, Kolmogorov 1932, Martin Löf 1996). Yet, the BHK interpretation by itself does not manage to explicate all the intuitive ideas that it contains. Consider the following two of its clauses:

> A proof of $A \to B$ is a *general method* of transforming any proof of A into a proof of B
>
> A proof of $\forall x A(x)$ is a *general method* of proving $A(a)$ for arbitrary a

Here, the reference to the notion of a "general method" leaves matters somewhat inexplicit, since it contains epistemic content that should have been brought forward in the exposition of the meanings of the logical constants (cf. Troelstra and van Dalen 1988). Still, relevant aspects of what we mean by a "general method" can be captured intuitively and developed into a list of desiderata that a formal semantics of 'traditional' intuitionistic logic has to fulfil.

I take the following four desiderata to be central:

[1] The label 'traditional' is used here, because there are motivations for using intuitionistic logic other than tracking provability, like modern considerations regarding indeterminacy (e.g. Dummett (1991) and Feferman (2014)), for which not all of the requirements listed below seem to be needed—or indeed even adequate. I will not, however, be concerned with these alternative motivations in this talk.

1. Our notion of a general method **should not verify the law of the excluded middle**. This should be obvious, but isn't in its entirety, as classical interpretations of the BHK clauses testify (cf. Sato 1997).

2. An **infinite conjunction of proofs should not count as a general method** to verify a universal statement—but any finite conjunction may. E.g., for each n we have a general (finite) method of deciding whether it is the sum of two primes (e.g. $\Box G(n)$ for all n), but this should not entail that we have a general method of deciding whether Goldbach's conjecture holds. This means that

$$\models G(n),\ \text{for each } n \quad \not\Rightarrow \quad \models \bigwedge_{n>0} G(n)$$

or, more explicitly:

$$\models \bigwedge_{n>0} \Box G(n) \quad \not\Rightarrow \quad \models \Box \bigwedge_{n>0} G(n)$$

3. The model **should account for objects being constructed, or even coming into existence**. This is idea we find most explicit in Brouwer (1981), but in others as well. In particular, then, the truth of a universal claim should *not* require or even entail the existence of *all* objects. Only insofar as an object has been constructed, should it be required to have the generalised property in question.

4. Finally, insofar as we are interested in modelling our epistemic prowess, the semantics **should satisfy the disjunction property,** i.e.

$$\models \varphi \vee \psi \quad \Rightarrow \quad \models \varphi \ \text{ or } \ \models \psi,$$

which the respective clause in the BHK interpretation explicitly states. Half a proof of φ and half a proof of ψ does not amount to a complete proof of $\varphi \vee \psi$.

These desiderata should be uncontroversial, yet none of the commonly known formal semantics satisfy all of them. Kripke models satisfy 1 and 3, and they satisfy 4 only if they are rooted, but they fail irreparably on 2. Topological semantics satisfy 1 and 2, but they don't satisfy 3 and 4. A combination of Beth and Kripke models satisfy 1, 2, and 3, but they, too, fail on 4. The failure of satisfying the disjunction property for topological and Beth models is due to the fact that disjunctions of formulas are interpreted as unions of sets or as collections of paths.

Part 2

Existing sheaf semantics (cf. Fourmann and Scott 1979, Troelstra and van Dalen 1988) satisfy 1, 2, and 3, but they also don't satisfy the disjunction property. Since these semantics interpreted formulas via open sets in the underlying topology of the sheaf, the reason for the failure of the disjunction property is the same as in the regular topological models. Contrary to this, my aim is to re-interpret the logical constants not as operations on the opens of the topology itself, but, following Melikhov (2015), on the sheaves defined over it. This allows for a way of interpreting $\varphi \vee \psi$ as a co-product of sheaves that yields the disjunction property. The resulting semantics, in brief, looks as follows:

Definition 1. Let X be a topological space. A *sheaf* over X is a pair (D, π), where D is a topological space, and $\pi : D \to X$ is a local homeomorphism.

A *section* is a function $\sigma : U \to X$ for an open $U \subset D$, s.t. $(\pi \restriction U)^{-1} \circ \sigma = id_U$. To illustrate this concept: Every section σ may stand for an object, and a $d \in U \subset D$ is taken to be its instantiation at $\sigma(d) \in X$. (If X is understood as a space of possible worlds, a section σ corresponds to the transworld-identity of an object.)

For a minimal example, I define product and co-product of sheaves:

Definition 2. Given a collection of sheaves (D_i, π_i), define their product as the map $\prod_i D_i \to X$ by setting

$$\vec{a} \in \prod_i D_i \qquad \text{iff} \qquad \pi_i(a_i) = \pi_j(a_j) \text{ for all } a_i \in D_i \text{ and } a_j \in D_j.$$

And their co-product as the map on the disjoint union $\bigsqcup_i D_i \to X$, by setting

$$(a, j) \in \bigsqcup_i D_i \text{ for each } D_j \text{ and } a \in D_j$$

Given a sheaf (D, π), define the *interpretation* of a relation symbol R as the subsheaf on $[\![R]\!] \subset D$ with $\pi \restriction_{[\![R]\!]}: [\![R]\!] \to X$. Let $\mathscr{D}$ be a domain of objects (with corresponding constants in the language). To each $d \in \mathscr{D}$ corresponds a section $\delta : U_d \to X$.[2] Then the formula $R(d)$ is interpreted as the subsheaf $([\![R]\!] \cap U_d, \pi \restriction_{[\![R]\!] \cap U_d})$. A formula is *valid* in the sheaf model, if its corresponding sheaf has a global section on X.

Now we can interpret the logical constants via operations on sheaves. For the relevant

[2] In other words, to each $d \in \mathscr{D}$ corresponds a set $U_d \subset D$, the instantiations of d in the worlds $\delta(U)$.

clauses, we have:

$$
\begin{aligned}
[\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \times [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \sqcup [\![\psi]\!] \\
[\![\varphi \to \psi]\!] &= Hom([\![\varphi]\!], [\![\psi]\!]) \\
[\![\forall x \varphi(x)]\!] &= \prod_{d \in \mathscr{D}} Hom(\delta, [\![\varphi(d)]\!])
\end{aligned}
$$

where $Hom(A, B)$ is the function sheaf from A to B.

This is the bare minimum that we need in order to see how the sheaf models satisfy the desiderata from part 1:

1. If the topology on X is connected, we have no non-trivial instances of LEM.
2. We can interpret the infinitary conjunction as $\prod_{i>0} [\![\varphi_i]\!]$, but if the topology on X is not alexandroff (i.e. not corresponding to a Kripke frame), then this product is not necessarily a sheaf.
3. Using the sections δ as an extent of the corresponding object d, we have introduced an existential guard into the interpretation of the universal quantifier, which accounts for objects 'coming into existence'.
4. Finally, as long as the sheaf is not disconnected, $[\![\varphi]\!] \sqcup [\![\psi]\!]$ only has a global section if either $[\![\varphi]\!]$ or $[\![\psi]\!]$ has.

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