THE ROLE OF PHYSICAL INTUITIONS IN MATHEMATICAL MODELS OF UNIFORM CHANCES

1. Introduction

An ongoing debate concerning the properties of Archimedean and non-Archimedean models of uniform chances is part of a broader discussion on philosophical positions with regard to the applicability of mathematics in the natural sciences. As a contribution to this debate, we apply Cartwright's analysis of scientific modeling to evaluate rival approaches to modeling uniform chances. We focus on the hypotheses which underlie such rival formalizations of an ideal uniform physical process that can be described as a fair spinner, and we argue that physical intuitions are not sufficient to exclude the use of non-Archimedean chances.

2. Regular chances in mathematical models of physical processes

A long-standing question in the foundations of probability theory concerns the role of events of zero chance. Such events typically arise when considering Archimedean probabilities (namely, probability measures that take values in Archimedean ordered fields, usually the field of real numbers) on infinite sample spaces. Relevant examples include gedankenexperiments such as uniform processes or of vast pluralities of trials, such as an infinite fair lottery or an infinite collection of coin tosses. In these cases, Archimedean probabilities are forced to assign zero chance to some events that can nevertheless occur. As a consequence, it is necessary to distinguish the notion of an impossible event from that of an event of zero chance.

However, these situations admit also models that assign a positive chance to each outcome that is considered possible. This property is called regularity. Early advocates of regularity include Carnap (1950), Kemeny (1955), Edwards et al. (1963), Shimony (1970), and Stalnaker (1970).

Since real-valued chances on infinite sample spaces cannot be regular, such regular models necessarily take values in a proper (and hence non-Archimedean) extension of the field of real numbers.

The use of regular non-Archimedean chances has been repeatedly criticized in the recent literature. Barrett [1], Parker [10], Pruss [11], and other authors argued that such models possess undesirable properties. For instance, Pruss seeks to argue that regular probabilities "carry more information than is determined by the plausible kinds of constraints on these probabilities" [11, p. 778].

Such critiques are sometimes anchored in purported 'physical' principles. For instance, Parker argues that "we should not require probabilities to be regular, for if we do, certain 'isomorphic' physical events (infinite sequences of coin flip outcomes) must have different probabilities, which is implausible" [10, Abstract]. Similar physical principles are invoked to rule out the use of regular chances for the modeling of a uniform process that is usually described as a fair spinner.

3. The fair spinner

Barrett describes the fair spinner as follows:

> [A] fair spinner [is] a perfectly sharp, nearly frictionless pointer mounted on a circular disk. Spin the pointer and eventually it comes to rest at some random point along the circumference.[1, p. 65]

Arguing from physical intuitions, Barrett, Parker, and Pruss claim that the mathematical models for this spinner should regard every point outcome as possible and should feature a high degree of symmetry, namely "invariance under rotations ... and modular translations" ([10], page 27) by an arbitrary real angle and real vector, respectively. From these considerations, they seek to conclude that physical principles rule out the use of regular chances.

Indeed, no known models of this process are simultaneously regular and possess such symmetries. From this consideration, Barrett, Parker, and Pruss conclude that physical principles force the choice of an Archimedean model based on the Lebesgue measure, that further requires identifying the set of possible outcomes with a real interval.

However, it is possible to model the fair spinner also with regular measures that are infinitesimally close to the Lebesgue measure [9]. Despite the incompatibility between regularity and invariance under real rotations and translations, some regular models feature limited forms of symmetry, such as invariance under rotations and modular translations by an arbitrary rational angle and rational vector, or invariance under such real transformations for a suitable subalgebra of subsets of the sample space [2, 3, 5].

4. An assessment of the models of the spinner

In order to assess the claim that the fair spinner cannot be meaningfully modeled by regular chances because they feature extrinsic properties and do not satisfy the physically-based hypothesis of invariance under real rotations and modular translations, we apply Cartwright's analysis of scientific modeling. Cartwright [6] introduces an important distinction in applying mathematics to analyze phenomena in science. Namely, a mathematical theory is not applied directly to such phenomena. Rather, one first builds a basic mathematical model of the phenomenon in question and then does one apply a full-fledged mathematical theory – not to the original phenomenon but rather to the basic mathematical model.

Thus, it is possible to distinguish three realms:

(a) pre-mathematical concepts best described as physical and/or naive-probabilistic phenomena, such as uniform processes, vast pluralities of trials etc.;
(b) basic mathematical models thereof;
(c) advanced mathematical tools brought to bear on the analysis of the basic mathematical models, such as the Lebesgue measure, non-Archimedean regular measures etc.

A pre-mathematical phenomenon typically admits multiple basic mathematical models, that in turn can be analyzed with distinct advanced tools.

Cartwright's analysis shows that it is necessary to keep distinct the pre-mathematical concepts from the corresponding basic mathematical models. Indeed, the conflation of a pre-mathematical concept with one of its basic mathematical models might lead to the unwarranted exclusion of other representations.

In the case of the fair spinner, the hypothesis of invariance under rotations of an arbitrary real angle is sufficient to rule out the current non-Archimedean models. However, this hypothesis belongs to the realm of the basic mathematical model based on identifying the sample space with a real interval, and not to the pre-mathematical concept. Failing to distinguish between these two realms led Barret, Parker, and Pruss to a circularity that predetermined the outcome of their analysis.

5. Physical intuitions do not rule out extrinsic features in mathematical models of the spinner

Cartwright's analysis of scientific modeling shows that the attack to regular probabilities from alleged physical principles is based upon circular reasoning. We further address the critique that regular chances introduce undesired extrinsic features in the model and show that it applies to Archimedean models as well.

Indeed, working in ZFC, no subset of the hypotheses on the spinner proposed by Barret is sufficient to single out a unique model, whether Archimedean or non-Archimedean. This result can be obtained from known properties of the real Lebesgue measure [4, 7, 8] and of hyperfinite counting measures of Robinson's nonstandard analysis [2, 3, 5].

Thus, the physical intuitions on this gedankenexperiment do not enable one to pin down a unique mathematical model, regardless of the hypothesis of regularity. In other words, the Archimedean models suffer from the same theoretical vice that has been attributed to regular models, namely carrying "more information than is determined by the plausible kinds of constraints on these probabilities" [11, p. 778].

6. Conclusions

Regular models of the fair spinner have been criticized because they do not satisfy physical intuitions and because they allegedly introduce undesirable extrinsic features. Cartwright's analysis of scientific modeling shows that both critiques are based upon circular reasoning; moreover, a careful mathematical analysis of the existing models of the spinner shows that the second critique applies to Archimedean models as well. As a consequence, physical intuitions are not sufficient to rule out the use of regular non-Archimedean chances.

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