Definitions and the Proof of Referentiality

In opening *Grundgesetze,* Frege undertakes in Part One an elaborately detailed stage setting for the proofs that are to constitute the logicist project which is his aim to present in Part Two. He does so by presenting the Begriffsschrift, his formal system, as a properly logical language, accordingly suited to be the milieu in which these proofs are to be conducted. That this is so, Frege shows by a demonstration that the Begriffsschrift is a fully referential language, a language, that is, in which every term, simple or complex, has a reference. This demonstration, carried through in §§29 − 31 of Volume 1 of *Grundgesetze*, is what has come to be known as Frege's Proof of Referentiality.

Why does Frege set himself this task? What value does he see in showing that the Begriffsschrift is fully referential? The answer that is forthcoming is that being fully referential qualifies the Begriffsschrift as a language properly suited for scientific applications, affording the deductive certainty required in such contexts. Frege's specific scientific concern is mathematical, to develop the "science of number" — arithmetic — as a formal realization of the ideal of a rigorous *scientific method for mathematics.* This is the logicist brief on which Frege thinks he has successfully delivered in *Grundgesetze*, saying with more than a pinch of hubris: "This ideal I believe I have now essentially achieved" (*Gg,* 44).

A language that is fully referential, and so properly suited for scientific tasks, Frege addresses as a *logically perfect* language. As Frege points out in *Grundgesetze*, and also in "On Sense and Reference," a logically perfect language is one in which *every* name, be it of a function or an object, is referential, bar none. This requires "that no new sign shall be introduced as a proper name without being secured a reference." Frege enshrines this notion in *Grundgesetze* by the following dictum that he entitles *the governing principle for definitions*: "Correctly formed names must always refer to something" (*Gg,* §28), where a correctly formed language consists of all and only correctly formed names. The purpose of the Proof of Referentiality is then to demonstrate that the Begriffsschrift satisfies this condition: that every expression of the Begriffsschrift is a referential name, either by being a primitive referential name, or by being composed of referential names. Accordingly, the Begriffsschrift is a correctly formed, referential language.

The meta-logical content of the proof of referentiality is well-known. If successful, it would constitute a consistency proof; the lesson of Russell's paradox is that it fails. Accordingly, the details of the proof, and wherein lies the cause of the inconsistency that dooms it, has been the topic of much careful and insightful investigation. This focus, however, has led to the presumption, responding to the question why Frege gives the proof of referentiality, that his reasons were primarily driven by meta-logical sensitivities: the purpose of the proof just was the demonstration of the consistency of the logical system. Without doubt, Frege recognized the meta-logical importance of his argument. But this does not get to the heart of the question. On our telling, Frege's primary purpose was not to offer a consistency proof *per se.* Rather, Frege's sensibilities primarily stemmed from a mathematical source, from a concern with how logic can carry mathematical content, with how logical propositions can be at the same time mathematical propositions. More specifically, the value Frege saw in the proof of referentiality is that it legitimized the efficacy of logical definitions *as definitions of concepts*, inclusive of definitions of the core notions of the mathematical project.

On Frege's conception, a language properly suited for scientific applications allows for the characterization of scientific concepts by means of definitions. Frege's view is that definitions can function fruitfully in this manner only when they introduce novel terms within a fully referential milieu, that is, within the context of a language in which every term is referential. The proof of referentiality is meant to show that the Begriffsschrift is such a language, and as such the proof is meant to lay the groundwork for the introduction of definitions.

Prima facie, for evidence we need look no further than how Frege situates the proof within the overall rhetorical structure of *Grundgesetze,* where it is explicitly placed in the discussion of definitions: the proof, encompassing §§29 − 31 of *Grundgesetze* I is located in a sub-part of Part One entitled "Definitions" under the heading "General Remarks". After an informal overview of the structure of the proof, we examine this positioning of the proof in the exposition in way of establishing its role in formulating the principles of definition. By these principles, definitions are explicit, and conservative over the language; *per* the Proof of Referentiality, the *definiendum* can only be abbreviatory of the *definiens.* Definitions can only be nominally creative; what they cannot do is introduce novel content.

But Frege's approach to definition raises questions, and his answers are the topic of the rest of this presentation. The key question is how can definitions be fruitful, and accordingly scientifically illuminating, if they are conservative over the logical language? In answering this question, a distinction is drawn between analytic definitions, justified by how they illuminate concepts, and proper definitions, justified by their utilization in proof. Fruitful definitions are those that are simultaneously analytic and proper. Our discussion will center on Definition Z of *Grundgesetze*, the definition of the number of a concept, as the prime example.

We turn next to Frege's approach to the definition of concepts, which is indirect, accomplished by their relation to their value-ranges as representative of concepts. The importance of this relation is that it allows for definitions to be specified as objectual identities; in Frege's logical language, there can be no well-formed assertions of identities of concepts. Critical to the reduction of concepts to their value-ranges is Definition A of *Grundgesetze.* This definition is intended to capture, in terms of a relationship between objects (more specifically 0-level entities of the conceptual hierarchy) and value-ranges, the predicativity of concepts (falling under). We refer to this relation as *membership*. An important aspect of this part of our discussion is how Frege's essay "On Concept and Object" is to be understood as part of the main dialectic of *Grundgesetze*.

The concluding discussion of the paper is the significance to Frege of Russell's Paradox. The paradox, as Frege himself notes, is a substitution instance of Theorem 1 of *Grundgesetze*, the assertion of the equivalence of predication and membership, which itself is an immediate consequence of Definition A and Basic Law V (in the right-to-left direction). What the paradox narrowly shows is that the proof of referentiality fails. But its broader significance is that the canons of definition collapse, as the logical transition from a concept to its value-range, necessary for the specification of definitions, fails. As Frege says in reflecting on the Russell's Paradox:

> Even now, I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not – at least conditionally – permissible to pass from a concept to its extension.

Thus, in the context of Frege's logicist program, the lesson of Russell's Paradox is that it undermines the definition of number, and hence the scientific content of the project.