Intrinsic Evidence for Large Cardinals and Structural Reflection

Abstract

Among new set-theoretic axioms, Large Cardinal Axioms hold a place of honour. These axioms have revealed to be immensely successful, both in terms of mathematical (set-theoretic) consequences, and in terms of connections they have helped establish among different areas of mostly, but not exclusively, set-theoretic interest.[1] The study of large cardinals seems to be, in addition, inexhaustible, as new hypotheses, whose usefulness may not yet be fully understood, but will presumably be revealed at some point, keep cropping up in set-theoretic work.

Their status and justification as new axioms of set theory is, on the contrary, a different story, as they do not seem to straightforwardly follow from the 'concept of set', and, thus, are not, according to a generally accepted classification of forms of justification for set-theoretic axioms, *intrinsically* justified.[2] However, it has been argued that, at least, 'small' large cardinals (those compatible with $V = L$) are intrinsically justified insofar as they are related to Closure and Reflection properties of V.[3]

In order to tackle the issue more systematically, in the paper we address all (fundamental) 'abstract motivating principles' which have been introduced (or invoked) to justify large cardinals.[4] These include Reflection, but also Resemblance and Uniformity (= Generalisation), crucially hinge on *elementary embeddings* (as already in Reinhardt's Reflection) and, taken jointly, prescribe increasingly stronger forms of self-similarity of V.[5]

More recently, further strengthenings of Reflection, such as [Welch, 2014]'s Global Reflection Principle (GRP) and [Bagaria, 2021]'s Structural Reflection Principles (SRP), have been formulated, which build on features of the aforementioned principles (in particular, on Resemblance and Uniformity), but could

[1] Cf. [Kanamori, 2009], especially the Introduction.

[2] For the origin of the classification, and its meaning, see [Gödel, 1947] and [Gödel, 1964]. Gödel's ideas are explained (and explored) in more detail in [Wang, 1996].

[3] Cf. [Tait, 1998], [Tait, 2005], [Koellner, 2009]). On Tait's (and, more generally, the 'bottom-up') strategy, see also [Barton, 2015] and [McCallum, 2021] discussing the principles in [Roberts, 2017].

[4] Some of these were identified by Gödel (cf. [Wang, 1974] and [Wang, 1996]); others are discussed in [Kanamori and Magidor, 1978], [Solovay et al., 1978], and [Maddy, 1988a], [Maddy, 1988b].

[5] For Reinhardt's theory, see, in particular, [Reinhardt, 1974a] and [Reinhardt, 1974b].

also be taken to be sub-principles of those main principles; moreover, both GRP and the strongest SRP's are equivalent to some of the strongest large cardinal hypotheses.[6]

Now, the main philosophical issue is whether, and in what sense, motivating principles, in particular, the more recent ones, are justified by the 'concept of set', taken to mean the 'iterative concept' (IC). In fact, as we argue in the paper, even Reflection, in its 'weak' form, may not be justified by IC, so the strongest principles, clearly, won't either.

But one could reason as follows. Let $\mathscr{C}$ be IC. One may take into account strengthenings of $\mathscr{C}$ through making further plausible ontological assumptions, such as: 'the Absolute exists', or 'predicative (impredicative) classes exist'. Now, if intrinsic justifiability is standardly taken to be equivalent to 'derivability from $\mathscr{C}$', now the notion could be extended to: 'derivability from $\mathscr{C}$ or its strengthenings'. In particular, one could view each of the strengthened concepts as expressing (further) conceptual content of $\mathscr{C}$, that is, as being directly implied ('conceptually grounded') by $\mathscr{C}$ itself, but one could also stipulate that the closer to $\mathscr{C}$ are the conceptual resources needed to justify a principle P the more intrinsically justified is P itself.

However, this strategy is far from being unexceptionable. In particular, its success crucially depends, among other things, on how one interprets class-theoretic discourse within set theory, and it is apparent that many mutually incompatible interpretations are available.[7]

Two further issues arise in connection with the (broadly conceived) justification of motivating principles. One could be called 'extension to inconsistency', and consists in the fact that the 'uncontrolled' iteration of a principle may lead to inconsistencies (as happened with Reinhardt's Reflection).[8] The other one is that some of these principles, such as Reflection, might not be able to justify all large cardinal hypotheses, but only some of them.

We conclude the paper by looking into where SRP locate themselves in this debate. In particular, one can tentatively say that, insofar as SRP:

- commit themselves only to definable classes;
- do not lead to a fateful form of 'extension to inconsistency'
- are flexible enough to produce variants which account for the existence of practically all large cardinals

SRP may be seen, in light of the way we have characterised the debate above, as being more justified, intrinsically, than the other principles, and also as enjoying a very robust extrinsic support.

[6]GRP implies that there are unboundedly many Woodin cardinals; the principle Π_1-SR, that is, Structural Reflection for Π_1-sentences, is equivalent to the existence of a supercompact cardinal.

[7]Cf., e.g, [Fujimoto, 2019].

[8]See [Koellner, 2009], p. 217.

References

[Bagaria, 2021] Bagaria, J. (2021). Large Cardinals as Principles of Structural Reflection. Pre-print.

[Barton, 2015] Barton, N. (2015). Richness and Reflection. *Philosophia Mathematica*, 24(3):330–59.

[Fujimoto, 2019] Fujimoto, K. (2019). Predicativism about classes. *The Journal of Philosophy*, 116(4):206–29.

[Gödel, 1947] Gödel, K. (1947). What is Cantor's Continuum Problem? *American Mathematical Monthly*, 54:515–525.

[Gödel, 1964] Gödel, K. (1964). What is Cantor's Continuum Problem? In Benacerraf, P. and Putnam, H., editors, *Philosophy of Mathematics. Selected Readings*, pages 470–85. Prentice-Hall.

[Kanamori, 2009] Kanamori, A. (2009). *The Higher Infinite*. Springer Verlag, Berlin.

[Kanamori and Magidor, 1978] Kanamori, A. and Magidor, M. (1978). The Evolution of Large Cardinal Axioms in Set Theory. In *Higher Set Theory*, pages 99–275. Springer Verlag, Berlin.

[Koellner, 2009] Koellner, P. (2009). On Reflection Principles. *Annals of Pure and Applied Logic*, 157(2-3):206–19.

[Maddy, 1988a] Maddy, P. (1988a). Believing the Axioms, I. *Bulletin of Symbolic Logic*, 53(2):481–511.

[Maddy, 1988b] Maddy, P. (1988b). Believing the Axioms, II. *Bulletin of Symbolic Logic*, 53(3):736–764.

[McCallum, 2021] McCallum, R. (2021). Intrinsic Justifications for Large-Cardinal Axioms. *Philosophia Mathematica*, 29(2):195–213.

[Reinhardt, 1974a] Reinhardt, W. N. (1974a). Remarks on reflection principles, large cardinals and elementary embeddings. In Jech, T., editor, *Proceedings of Symposia in Pure Mathematics*, volume XIII, 2, pages 189–205. American Mathematical Society, Providence (Rhode Island).

[Reinhardt, 1974b] Reinhardt, W. N. (1974b). Set Existence Principles of Shoenfield, Ackermann and Powell. *Fundamenta Mathematicae*, 84:5–34.

[Roberts, 2017] Roberts, S. (2017). A strong reflection principle. *The Review of Symbolic Logic*, 10(4):651–62.

[Solovay et al., 1978] Solovay, R., Reinhardt, W., and Kanamori, A. (1978). Strong Axioms of Infinity and Elementary Embeddings. *Annals of Mathematical Logic*, 13:73–116.

[Tait, 2005] Tait, W. (2005). Constructing cardinals from below. In Tait, W., editor, *The Provenance of Pure Reason. Essays in the Philosophy of Mathematics and its History*, pages 133–54. Oxford University Press, New York.

[Tait, 1998] Tait, W. W. (1998). Zermelo's Conception of Set Theory and Reflection Principles. In Schirn, M., editor, *Philosophy of Mathematics Today*, pages 469–483. Clarendon Press, Oxford Press, Oxford.

[Wang, 1974] Wang, H. (1974). *From Mathematics to Philosophy*. Routledge & Kegan Paul, London.

[Wang, 1996] Wang, H. (1996). *A Logical Journey*. MIT Press, Cambridge (MA).

[Welch, 2014] Welch, P. (2014). On Global Reflection Principles. Currently in the Isaac Newton Institute pre-print series, No. NI12051-SAS.